

Initial State Independent Equilibration at the Breakdown of the Eigenstate Thermalization Hypothesis

Abdellah Khodja,^{1,*} Daniel Schmidtke,^{1,1,†} and Jochen Gemmer^{1,‡}

¹*Fachbereich Physik, Universität Osnabrück, Barbarastrasse 7, D-49069 Osnabrück, Germany*

This work aims at understanding the interplay between the Eigenstate Thermalization Hypothesis (ETH), initial state independent equilibration and quantum chaos in systems that do not have a direct classical counterpart. It is based on numerical investigations of asymmetric Heisenberg spin ladders with varied interaction strengths between the legs, i.e., along the rungs. The relaxation of the energy difference between the legs is investigated. Two different parameters, both intended to quantify the degree of accordance with the ETH, are computed. Both indicate violation of the ETH at large interaction strengths but at different thresholds. Indeed the energy difference is found not to relax independently of its initial value above some critical interaction strength which coincides with one of the thresholds. At the same point the level statistics shift from Poisson-type to Wigner-type. Hence the system may be considered to become integrable again in the strong interaction limit.

PACS numbers: 03.65.Yz, 75.10.Jm,

I. INTRODUCTION

Even in closed quantum systems one may observe relaxation, in spite of time evolution being generated by unitary operations. This statement has many aspects, it may imply that a reduced density matrix of a part of some system approaches a thermal state [3, 4, 7, 42], it may imply that expectation values of specific observables evolve more or less against constant values [1, 43], or it may additionally even imply that these constant values do not depend on the initial state [1, 17, 44]. It is this latter initial state independence (ISI) which is in the focus of the paper at hand. Among the concepts addressing this issue is the so called eigenstate thermalization hypothesis (ETH) [9, 10]. According to the ETH, the expectation values of a typical observable \hat{D} as computed from energy eigenstates $|n\rangle$ should be a smooth function of energy E_n , i.e., $\langle n|\hat{D}|n\rangle \approx \langle n'|\hat{D}|n'\rangle = \bar{D}_{eq}(E)$ if $E_n \approx E_{n'} \approx E$. As is well known, if the ETH applies, the long time averages of all expectation value $\text{Tr}\{\hat{\rho}\hat{D}(t)\}$ corresponding to any initial state $\hat{\rho}$ that lives inside some energy shell, are equal [36]. Furthermore, if the spectrum of the Hamiltonian \hat{H} is “non-resonant” (roughly speaking: any energy difference occurs only once) and the ρ_{nn} are fairly spread over the energy shell, then the actual $\text{Tr}\{\hat{\rho}\hat{D}(t)\}$ deviates from its long time average very rarely [6]. Hence, the three features, i. ETH agreement, ii. non-resonant spectrum and iii. dilute eigenstate occupation, guarantee the initial state independent (ISI) relaxation of the expectation value towards a specific “equilibrium” value. Since this behavior is observed and expected for practically all physical relaxation phenomena, the question arises whether the above three features apply to all those

situations. This question has been approached from the perspective of quantum chaos, and various rigorous results exist that suggest i. and ii. apply to systems which have a direct classical counterpart which is chaotic [17]. Much less is known for systems that do not have a direct classical counterpart [10]. However, while the three features are sufficient for ISI, they are not necessary in a mathematical sense: classes of initial states exist that exhibit ISI even though the ETH may not apply. Some papers put much more emphasis on the extremely high relative frequency with which ISI may be expected if initial state are drawn essentially at random from some prescribed sets, rather than on the ETH [5, 7, 43, 46]. However, it may be the case that the relative frequency of initial states from the above sets, that exhibit a significant deviation of the respective expectation value from its equilibrium value at all, even at $t = 0$, is very low. In this case results of relative-frequency-type would imply the existence of a majority of states the expectation values of which start and remain in equilibrium. However, no conclusions on state, that actually do exhibit non-equilibrium expectation values in the beginning, could be drawn.

In view of this, a class of initial states that are specifically tailored to exhibit largely deviating expectation values, but live in narrow energy shells at the same time has recently been suggested [12]. These states (which will be explained in detail below) have been termed microcanonical observable displaced (MOD) states. MOD states are “natural” in the sense that they may be viewed as results of state determination according to Jayne’s principle under the conditions of a given expectation value and the state living in an certain energy shell. For the remainder of this paper we focus on dynamics as resulting from such initial MOD states.

In order to set the content of the paper at hand in context to the state of research, we need to specify our

*Electronic address: akhodja@uos.de

†Electronic address: danischm@uos.de

‡Electronic address: jgemmer@uos.de

notion of ETH in somewhat more detail. Consider

$$\bar{D} = \sum_{n=1}^d p_n D_{nn}, \quad \Sigma^2 = \sum_{n=1}^d p_n D_{nn}^2 - \bar{D}^2, \quad (1)$$

where D_{nn} are diagonal matrix elements with respect to the Hamiltonian eigenstates $|n\rangle$ with eigenvalues E_n , $p_n \propto e^{-(E_n - \bar{E})^2 / 2\sigma_E^2}$ is a probability distribution centered at \bar{E} , and d is the Hilbert-space dimension. The quantities $\bar{D}(\bar{E}, \sigma_E)$ and $\Sigma(\bar{E}, \sigma_E)$ are obviously functions of \bar{E} and an energy width σ_E . Routinely, the ETH is said to be fulfilled if Σ is small for small σ_E . Technically, ISI equilibration is only guaranteed for all initial states from a given energy window σ_E at \bar{E} iff $\Sigma = 0$, [37]. If instead a finite-size scaling scheme is employed, i.e., $\Sigma \rightarrow 0$ for $N \rightarrow \infty$ (N being the “system-size”, e.g., the number of spins, etc.) not only the Hamiltonian, but also the observable \hat{D} often scales with N in some fashion as well (cf. below). In this context it is relevant to note that ISI equilibration is only ensured if Σ itself approaches zero in the limit of large system sizes, the vanishing of, e.g., Σ^2/N , is not sufficient. However, in our analysis we do not intend to demonstrate ISI equilibration for all initial states of a given energy window but rather focus on the afore mentioned special class of states, i.e., MOD states. Below we will employ a quantifier which is closely related to Σ but independent of the scaling of the observable itself.

In order to put our work into perspective, we list some exemplary results from the literature on the generic scaling properties of Σ , and classify our results in relation to these. In the context of translational invariant, solid-state type observables and models, the observable is frequently defined to scale “extensively”, like, e.g., a current, a total kinetic energy, etc. In this case, there is substantial numerical evidence that Σ scales more or less as $\Sigma \propto d_{eff}^{-\gamma}$, where d_{eff} is the effective dimension, i.e., the number of states within the respective energy shell and γ is some constant [22, 38, 45]. Note, however, that the scaling behavior of Σ generally depends strongly on whether the observable itself is defined to scale extensively, intensively or else. Since the effective dimension d_{eff} usually increases (exponentially) with system size, Σ tends to zero with increasing system size whenever $\gamma > 0$. The existing literature contains numerous examples exhibiting $0 \leq \gamma \leq 1/2$ for few body observables. $\gamma = 0$ appears to be related to integrable systems, $\gamma = 1/2$ for fully chaotic systems, where both definitions may vary depending on the respective authors [47].

In Ref. [38], however, also examples are discussed with $\gamma < 0$ for extensive observables, i.e., the unscaled Σ itself increases with system size. This occurs for spin and energy currents in the (integrable) Heisenberg chain at some parameter regime. In contrast to that, in Ref. [12] the same Heisenberg chain model is investigated while the considered observable is different, namely the energy difference between two parts of the chain. The latter observable is also extensive. Nonetheless, in this case

$\Sigma = \text{const}$, i.e., $\gamma = 0$ is found. As already mentioned above, for $\gamma \leq 0$ ISI equilibration is not guaranteed, i.e., the construction of initial states for which the expectation value of \hat{D} does not decay towards the corresponding equilibrium ensemble value is always possible. Thus we will primarily focus on the question whether such a “incomplete decaying” occurs for MOD states for models and observables exhibiting $\gamma \leq 0$.

For the remainder of this paper we follow [12] in investigating the (extensive) energy difference between certain system sections. We present a model that allows for a continuous tuning of the scaling parameter γ all the way from $\gamma > 0$ to $\gamma < 0$, i.e., depending on some system parameter, Σ either decreases, remains constant or increases under upscaling of the system-size. We also follow [12] in considering an alternative parameter v for the “prediction” of ISI for MOD states:

$$v = \left(\frac{\Sigma}{\delta_D} \right)^2 \quad (2)$$

δ_D^2 denotes the spectral variance of the observable \hat{D} in the respective energy shell, i.e., $\delta_D^2 = \sum_{n=1}^d p_n (\hat{D}^2)_{nn} - (\bar{D})^2$ (the bars over D symbolize the averaging as defined in Eq. (1)). Note that v is, other than Σ , dimensionless. Thus for the scaling behavior of v , the scaling behavior of the observable itself (extensive, intensive or else) is irrelevant. It has been proposed that ISI for MOD states occurs, if and only if $v \rightarrow 0$ in the limit of large systems. Although closely related to the ETH, the v criterion is not quite the same, for it may possibly approach zero even if Σ increases, if δ_D^2 increases more quickly.

The present paper is organized as follow: In section II, we shortly introduce the investigated models and the addressed observable, i.e., the energy difference. In section III, we present the results on Σ, v in dependence of the tuning parameter κ and show the existence of two different regimes based on the scaling behavior of the former. The computations in Sec. IV clarify whether MOD states give rise to ISI relaxation of the energy difference and how this relates to the results of Sec. III. We discuss the integrability of the model at hand in Sec. V using the generalized Brody parameter, whose behavior turns to be correlated to the ETH parameter v . Finally, we close with summary and conclusion.

II. MODELS AND OBSERVABLES

The model we address in this work is an asymmetrical Heisenberg-ladder which consists of two XXZ spin-1/2 chains of different length, coupled along some “rungs” with interaction strength κ , cf. Fig. 1. The total Hamilton operator is given as follow

$$\hat{H} = \hat{H}_L + \hat{H}_R + \kappa \hat{H}_I \quad (3)$$

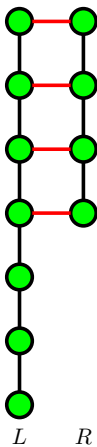


Figure 1: Schematic visualization of the investigated spin-ladder. To suppress symmetry, the number of spin in the right chain is taken to be different than the left chain in the following systematic fashion $N_L = 2N_R - 1$. Note that both chains only interact by vertically opposing sites (here indicated by red rungs).

where

$$\hat{H}_{L,R} = \sum_{i=1}^{N_L, N_R} [(\hat{S}_i^x \hat{S}_{i+1}^x + \hat{S}_i^y \hat{S}_{i+1}^y) + \Delta \hat{S}_i^z \hat{S}_{i+1}^z] \quad (4)$$

describes the two side-chains. Δ denotes the anisotropy parameter, which is kept at $\Delta = 0.1$ throughout the entire investigation. N_L, N_R are the numbers of spins with respect to left and right spin chain respectively, thus the total number of spins is $N = N_L + N_R$. We choose $N_L = 2N_R - 1$ in order to suppress symmetry, since for the symmetrical case the ETH is trivially fulfilled. The two subsystems are allowed to exchange energy through the interaction Hamiltonian, which reads

$$\hat{H}_I = \sum_{i=1}^{N_L} [(\hat{S}_i^{x,L} \hat{S}_i^{x,R} + \hat{S}_i^{y,L} \hat{S}_i^{y,R}) + \Delta \hat{S}_i^{z,L} \hat{S}_i^{z,R}]$$

Obviously, at any non-zero κ , this model is not accessible by a Bethe ansatz. Thus, in this sense it is always non-integrable. For a more detailed discussion of integrability, see Sec. V

The observable we are going to investigate is the energy difference operator $\hat{D} = \hat{H}_L - \hat{H}_R$. Our motivation for this particular choice stems from the intuitive example of two cups of coffee brought into contact with each other, where one anticipates more or less a uniform energy density throughout the system. The model (3) has been numerically studied in detail for weak interactions, where the ETH turned out to be valid [12],

III. COMPUTATION OF ETH-QUANTIFYING PARAMETERS

Numerical computation of ETH related data like Σ , \bar{D} , etc. for large systems is not a trivial task. Usually, it requires exact diagonalization [11, 12, 38] that is limited to rather small system sizes. Thus, we apply a recently suggested method [18] that is based on dynamical typicality and allows for the extraction of information about ETH from the temporal propagation of pure states. This propagation can be performed by iterative algorithms such as Runge-Kutta [18, 20, 21], Chebyshev [23, 24] etc. and is feasible for larger system sizes. We use a Chebyshev iterator with reasonably small time step in order to improve the computation accuracy. Due to typicality-related reasons, the so-computed quantities \bar{D} and Σ are subjected to statistical errors. These errors, however, turn out to be small. Apart from the model size $N = 14$, which is treated by exact diagonalization (LAPACK-routine), all data in this section has been computed using the above technique. This way we are able to address systems up to $N = 26$.

We focus on a narrow energy shell around $E = 0$ which is the energy regime with the highest density of states. More precisely we choose $\bar{E} = 0$ and $\sigma_E = 0.6$. To set this into perspective, the total energy range of this type of model is on the order of N . The results for the ETH-fluctuations Σ are depicted in figure 2(a).

Figure 2(a) indicates that there are two clearly distinct regimes: Above $\kappa = 3.7$ a convergence of the ETH parameter Σ to zero appears unlikely, even though the presented data may not allow for the precise determination of Σ in the large system limit. At $\kappa = 3.7$ a simple linear scaling also indicates a non-zero Σ in the large system limit, even though the presented data may not be entirely conclusive. However, below $\kappa = 3.5$ Fig. 2(a) clearly indicates $\Sigma \rightarrow 0$ for $N \rightarrow \infty$. Thus for $\kappa \geq 3.7$ the considered model potentially shows no ISI relaxation of the energy differences between the two chains. This is to be contrasted with the overall concept of heat conduction or the second law of thermodynamics, which demands that energy eventually distributes evenly over all parts of a system, regardless of how uneven it was distributed in the beginning.

Figure 2(b) displays the ETH parameter v which has been suggested as a “detector” of ISI relaxation [12]. While the behavior of Σ and v may appear similar at first sight, there are relevant differences: for all $\kappa \leq 3.7$ v clearly vanishes in the limit of large systems, whereas at $\kappa = 4.0$ a more or less constant scaling occurs for $N \geq 20$. Thus the “switching” from vanishing to non-vanishing values in the large system limit appears to occur at $\kappa \approx 3.7$ for Σ , whereas it occurs at $\kappa \approx 4.0$ for v .

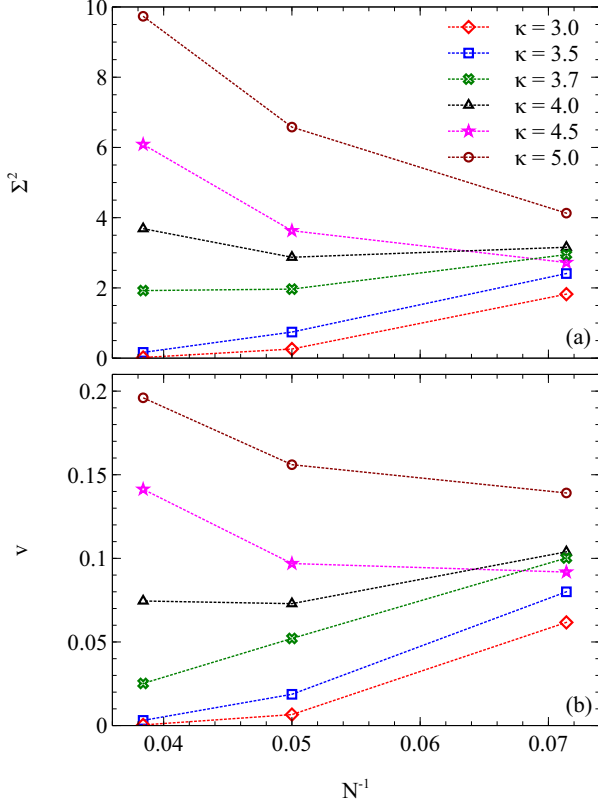


Figure 2: Comparison of the finite size scalings of the "bare" ETH-fluctuation Σ (a) and the scaled ETH parameter v (b) in dependence of the interaction strength κ . Both quantifiers vanish in the large system limit for small κ while they reach constant values (or possibly even diverge) for large κ . For Σ the transition between the two different regimes appears to occur at $\kappa \approx 3.7$. Whereas for v the transition appears to occur at $\kappa \approx 4.0$.

IV. INVESTIGATION OF ISI-RELAXATION OF ENERGY DIFFERENCES FROM INITIAL MOD-STATES

In this section we are going to discuss the long time behavior of $\langle D(t) \rangle$ as resulting from initial states from the class of MOD states. The latter are defined as follows:

$$\rho_{\text{MOD}} \propto e^{-(\hat{H}-H_0\hat{1})^2-\beta^2(\hat{D}-D_0\hat{1})^2}/2\sigma^2. \quad (5)$$

These states may be viewed as being based on Jayne's principle: They represent the maximum-entropy states under given means and variances for the energy and the observable. Since \hat{H} and \hat{D} do not commute with each other, it is not possible to generate states with arbitrary values of total energy $\langle \hat{H} \rangle_{\text{MOD}}$, energy difference $\langle \hat{D} \rangle_{\text{MOD}}$ and the respective variances. However, by tuning the parameters H_0, D_0, β, σ , carefully, we are able to prepare initial states having a well-defined energy width $\sigma_E \approx 0.6$ around $E = 0$ and exhibiting initial expectation values for the observable $D(0) = \langle \hat{D} \rangle_{\text{MOD}}$, which deviate strongly from their corresponding equilibrium values

D_{eq} ; more quantitatively, $\langle D \rangle_{\text{MOD}}$ reaches about 50% of the difference between its highest possible value and its long-time average. To enable such strong deviations we choose $D_0 = \pm(N_L - 2)$ throughout all investigations. This is to be contrasted with various quench scenarios [25, 26], where the initial deviations from the equilibrium value are rather small. Note also that such MOD states do not necessarily feature a smooth probability distribution with respect to the corresponding energy eigenbasis, as required by Ref. [15] in order to establish ISI.

Neither constructing states of the MOD-type (5) nor propagating such states according to the Schrödinger equation is numerically simple for larger systems. Thus, we instead prepare and evolve a corresponding pure state $|\phi_{\text{MOD}}\rangle$, which exhibits, up to very small statistical errors the same $\langle D(t) \rangle$ as ρ_{MOD} :

$$|\phi_{\text{MOD}}\rangle = \langle \varphi | \rho_{\text{MOD}} | \varphi \rangle^{-1/2} \rho_{\text{MOD}}^{1/2} | \varphi \rangle, \quad (6)$$

where $|\varphi\rangle$ is a random state drawn according to the unitary invariant (Haar-) measure. This concept relies on dynamical typicality and has been explained and applied in e.g., [12, 20, 34, 38].

As an example the time evolutions of $\langle D(t) \rangle$ for $D(0) = \pm 7, N = 26$ and two different interaction strengths, namely $\kappa = 3, 4.5$, are displayed in Fig. 3 First

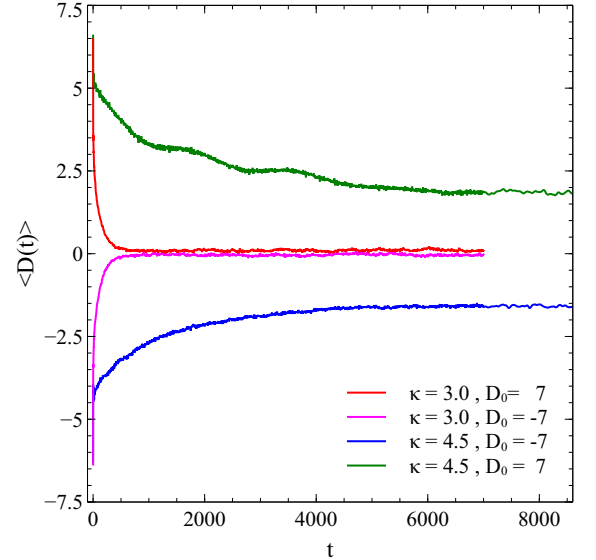


Figure 3: Time evolution of $\langle D(t) \rangle_{\text{MOD}}$ for various interaction strengths starting at $D_0 = \pm 7$ for systems of size $N = 26$. In case of $\kappa = 3.0$, ISI equilibration is obvious but for $\kappa = 4.5$ the expectation value converges against finite offsets indicating that some portion of the initial value persists (see text for details). Note that due to slightly different $|D(0)|$ occurring for $\kappa = 4.5$ the "equilibrium values" for corresponding curves also slightly differ.

of all one should note that the non resonance condition and the dilute eigenstate occupation principle obviously apply since in all instances $\langle D(t) \rangle$ converges against constant values with time. However, while the energy dif-

ference $\langle D(t) \rangle$ clearly vanishes quickly for $\kappa = 3.0$ (as required by the validity of the ETH) this is not the case at $\kappa = 4.5$. There it appears that a fixed portion of the initial value persists. Hence, this system indeed partially preserves an uneven distribution of local energy in the long time limit, which may be viewed to be at odds with the principle of heat conduction. This finding motivates the introduction of the “ISI quantifying” parameter Λ , which we define as the long time equilibrium value divided by the initial expectation value. Since the preserved portion seems to be more or less independent of actual D_0 , we calculated only dynamics featuring $D_0 = N_L - 2$. Thus, the definition of the “ISI quantifying” parameter reads

$$\Lambda = \frac{\langle \phi_{\text{MOD}} | D(t) | \phi_{\text{MOD}} \rangle}{\langle \phi_{\text{MOD}} | D(0) | \phi_{\text{MOD}} \rangle} \quad \text{with } t \gg \tau, \quad (7)$$

where τ is the relaxation time. Obviously $\Lambda = 0$ indicates ISI whereas larger Λ indicate a violation of ISI. Furthermore, note that Λ , like v , is insensible with respect to scaling of observables with system size and hence applies to any kind of observable. We computed Λ for three different interaction strengths κ for increasing system sizes. The result is displayed in Fig. 4. Clearly Λ vanishes in

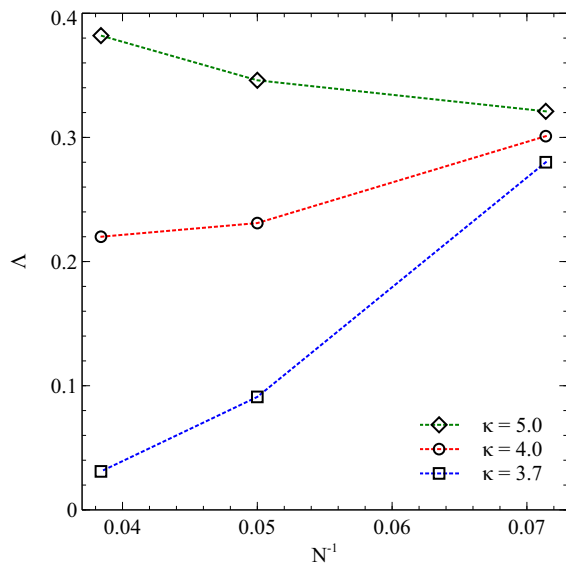


Figure 4: Scaling of Λ with various interaction strengths and system sizes. At $\kappa = 3.7$ the ISI-quantifying Λ tends to zero indicating ISI equilibration whereas for both larger interaction strengths Λ remains finite in the limit of large systems quantifying deviations from the equilibrium value D_{eq} . This behavior corresponds rather accurately to the finite size scaling behavior of v , cf. Fig. 1. Statistical errors are of order of symbol size.

the limit of large system for $\kappa = 3.7$, thus ISI relaxation holds at this interaction strength. At and above $\kappa = 4$, Λ appears to converge against non-zero values, hence no ISI relaxation exists at these interaction strengths, not

even for $N \rightarrow \infty$. This result should be discussed in relation to the results of Sec. III. Obviously the transition from ISI to non-ISI with increasing interaction strength happens at the same point at which v starts to converge against non-zero values. Thus the behavior of v predicts ISI rather precisely. Σ , on the other hand, starts to converge against non-zero values at $\kappa > 3.5$, i.e., in a regime in which ISI is still present. Thus, in this sense, v appears to be a more reliable predictor of ISI than Σ . This is one of the main results of this paper.

V. INTEGRABILITY INVESTIGATIONS

So far, we focused on the issue of ISI equilibration of a specific observable and found the existence of two regimes, ISI and non-ISI, depending on the interaction strength. Next we address the existence of an integrable regime in the model at hand and study its relevance for the emergence of ISI for the MOD states. Again, similar to the discussion of ETH and ISI, a range of papers more or less explicitly states that non-integrability is imperative for ISI [14], whereas other works analyze ISI without even mentioning integrability. Also different features of “statistical relaxation” (other than ISI) are addressed; examples exist in which the occurrence of statistical relaxation does not depend on integrability [19]. However, this type of investigation generally suffers from a conceptual shortcoming: there is no unambiguous definition for integrability in quantum mechanics. Contrary to classical mechanics notions like phase space, Lyapunov exponents and ergodicity are not well defined. In the context of lattice particle systems, quantum integrability is sometimes associated with being accessible by the Bethe ansatz, i.e., the 1D quantum Heisenberg-chain with nearest neighbor interaction [29], the 1-D δ -function interacting Bose [30] and the Fermi [31] gases are considered as integrable. According to this later definition the model considered in this paper is integrable. Nevertheless, the break down of the ETH for large interaction strength may be viewed as being due to the proximity of the integrable (according to any standard definition) limit of non-interacting spin-dimers. In order to address and quantify this possible integrability, we resort to the well known approach which is based on the Nearest Neighbor Level Spacing Distribution (NNSD) denoted by $P(\Delta\epsilon)$ [39, 40]. The distinction between integrability and chaos is as follows: if the most frequent energy spacing is approaching zero $\Delta\epsilon = 0$ and the shape of the NNSD mimics a Poisson distribution, the system is considered to be integrable. Whereas, if the most frequent energy spacing takes some finite value (level repulsion) with an NNSD shape of Wigner-Dyson type, the system is considered to be non-integrable. This classification of quantum systems using level statistics has been derived in the context of quantum models, whose corresponding classical counterparts are chaotic [32] and has been

adopted even for quantum systems which do not have classical counterpart, e.g., spin systems.

For most finite systems of condensed matter type the NNSD turns out not to correspond exactly neither to Poisson nor Wigner-Dyson like distributions. To deal with intermediate statistics, Brody proposed in Ref. [33] to compare each real NNSD to a one parameter ω -family of analytically given NNSD's, where $\omega = 0$ corresponds to pure Poisson and $\omega = 1$ to pure Wigner statistics. Matching a real NNSD to a pertinent Brody-NNSD thus yields a specific ω that may be used to quantify the closeness to either Poisson or Wigner, respectively. Following this scheme, we computed (by means of exact diagonalization) and normalized an NNSD for various interaction strengths and $N_R = 6$ in a narrow energy interval around $E = 0$. The results, for $\kappa = 0.3, 4.0$, together with the matching Brody NNSD's are displayed in Fig. 5. Obviously the agreement is rather good. This justifies

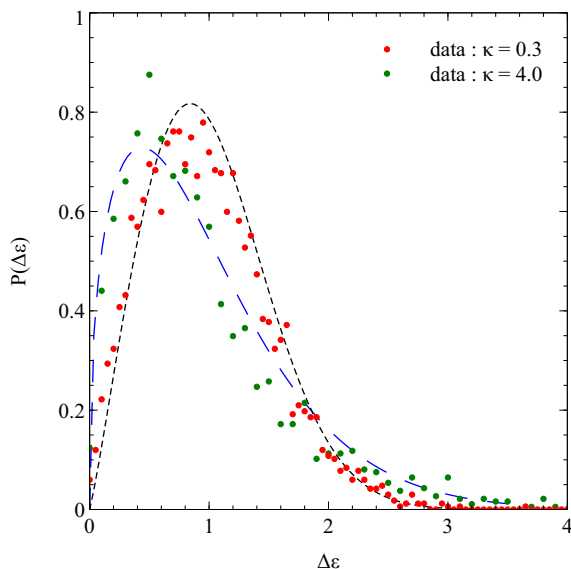


Figure 5: NNSD for $\kappa = 0.3$ and $\kappa = 4.0$ both for $N = 14$ where symbols display computed data and dashed lines corresponding Brody distributions. In case of $\kappa = 0.3$ the Brody parameter reads $\omega \approx 1.1$ (Wigner-Dyson type) and for $\kappa = 4.0$ the Brody parameter reads $\omega = 0.4$ (Poisson type).

the usage of the above described method to quantify the “degree of integrability” by means of the parameter ω . Finally, Fig. 6 displays ω as a function of κ together with the ETH parameter v . First one should note that for large interaction strengths the NNSD is much closer to Poisson than to Wigner. This indicates that this model class may indeed become integrable again for stronger interactions, say $\kappa \approx 3.7$, which are nonetheless far away from the integrable dimer limit at $\kappa \gg 1$. To repeat, this integrability is not induced by the possibility of applying a Bethe ansatz. We are furthermore unable to judge whether this integrability may be explained by stretching the quantum KAM theorem [48] all the way down

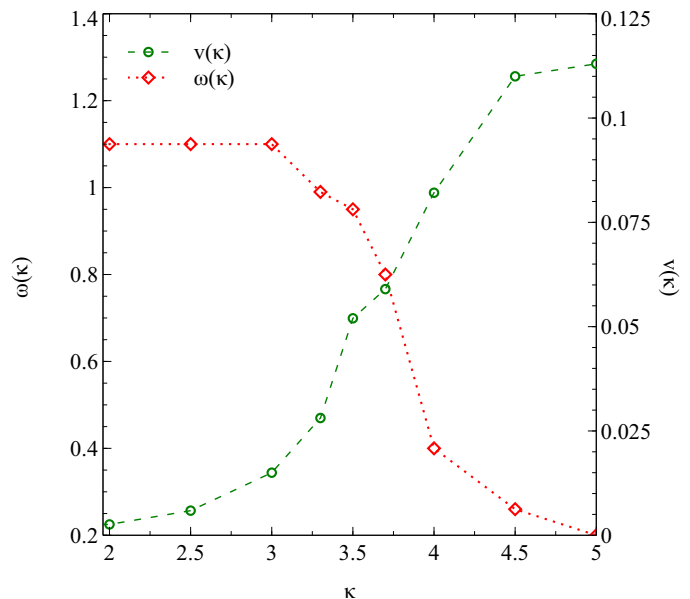


Figure 6: Comparison of Brody parameter ω and alternative ETH parameter v . The transition from integrable to non-integrable systems indicated by ω obviously coincides with the transition from the ISI to non-ISI regime, respectively. This links the equilibration dynamics to quantum chaos; see text for details. Note that ω is gained from systems of $N = 14$ whereas v is gained from systems of size $N = 20$.

from $\kappa \rightarrow \infty$ to $\kappa \approx 3.7$. However, it is striking that the transition from ISI to non-ISI (indicated by increasing v) happens at the very same point at which the NNSD changes from Wigner to Poisson (indicated by decreasing ω). While this finding is just based on numerics it suggests that “chaoticity” in the sense of a large Brody parameter may indeed be a sufficient criterion for the ISI relaxation of few-body observables for systems starting in MOD states. Investigations on different spin systems that point in a similar direction [38] also exist.

VI. TOWARDS THE PHYSICS BEHIND THE NUMERICAL FINDINGS

While the numerical results clearly indicate a breakdown of “chaoticity” as well as of full relaxation of differences of local energies at strong couplings, the physical reason for this behavior is yet unclear. In the following we shortly speculate about such physical reasons, thus arriving at some suggestions for further research. All our findings refer to exchange of local energy between two asymmetric legs of a ladder. It may, however, be elucidating to consider the exchange of local energy between the part of the system that is a regular ladder (upper part in Fig. 1) and the part that consists of the “elongation” of one leg that really is just a chain (lower part in Fig. 1). If energy exchange between these two parts (ladder and chain) is suppressed, this will result

in a suppression of energy exchange between the original asymmetric legs, since the two energy-differences surely have an overlap in the sense of a Mazur inequality [49]. Thus taking the “ladder and chain” point of view, two features are evident: In the limit of large κ the mean level spacing in the ladder will eventually become significantly larger than the mean level spacing in the chain. Furthermore, the chain is integrable in the sense of a Bethe-Ansatz, whereas the ladder is not. Regarding the increasing level spacing in the chain, it may be the case that transitions that amount to an exchange of local energy become more and more off-resonant. If the coupling strength (here: between ladder and chain, i.e., not κ) remains constant but the coupling becomes rather off-resonant, the relaxation of the respective observable may eventually not only be slowed down but inhibited completely [50]. Concerning the integrability, the eigenstates of the chain may in principle be described by a respective set of rapidities, whereas this is not applicable to the eigenstates of the ladder. This structural difference between the eigenstates may cause the eigenstates of the chain to scatter strongly at the interface, thus preventing them from penetrating deeply into the ladder. This could also lead to an effective suppression of the transitions that facilitate an exchange of local energy. Deciding which of the two above schemes is (if at all) dominantly responsible for the inhibition of the exchange of local energy is beyond the scope of the present paper. Future research, however, could focus on the “ladder and chain partition” and take more variables, other than just local energy differences into account.

VII. SUMMARY AND CONCLUSION

The paper at hand aims at clarifying the interrelations between the eigenstate thermalization hypothesis (ETH),

initial state independent relaxation and chaos in quantum systems. The investigations are of primarily numerical character and focus on a class of (asymmetric) ladder type spin systems with variable interaction strength between the legs. We investigated the energy difference between the legs of the spin system where we found that two ETH quantifying parameters, a standard one and an alternative, recently suggested one, indicate violation of the ETH, even in the limit of large systems above certain “threshold” interaction strengths. However, the thresholds differ for the standard and the alternative ETH parameter.

Furthermore, the relaxation behavior of the energy difference between the legs of the spin system is analyzed for a specific class of initial states. It is found that those energy differences no longer equilibrate to zero above a certain interaction strength. This interaction strength precisely coincides with the ETH violation threshold of the alternative ETH parameter but not with the threshold of the standard one.

Finally the level spacing statistics are considered. It turns out that they shift from Poisson-type to Wigner-Dyson type with increasing interaction strengths, again rather distinctly at the threshold of the alternative ETH parameter. To conclude: numerical evidence suggests that the alternative ETH parameter reliably signals the relaxation of an observable towards a common value that is independent of the initial, possibly largely off-equilibrium value of the respective observable. Moreover there appears to be a strong correlation of this alternative parameter with either chaotic or integrable type of level statistics.

-
- [1] J. von Neumann, *Beweis des Ergodensatzes und des H-theorems in der neuen Mechanik* Zeitschrift für Physik **57**, 30-70 (1929)
 - [2] J. Gemmer, M. Michel, and G. Mahler, *Quantum Thermodynamics: Emergence of Thermodynamic Behavior within Composite Quantum Systems*, Lect. Notes Phys. **657**, 2nd edition (Springer, Berlin, 2009)
 - [3] S. Popescu, A. J. Short and A. Winter, Nature Physics **2**, 754 - 758 (2006).
 - [4] S. Goldstein, J.L. Lebowitz, R. Tumulka, and N. Zanghi, Phys. Rev. Lett **96**, 050403 (2006)
 - [5] N. Linden, S. Popescu, A. J. Short, and A. Winter, Phys. Rev. E **79**, 061103 (2009).
 - [6] P. Reimann, Phys. Rev. Lett. **101**, 190403 (2008).
 - [7] O. Lychkovski, Phys. Rev. E **82**, 011123 (2010).
 - [8] A. Riera, C. Gogolin, and J. Eisert, Phys. Rev. Lett. **108**, 080402 (2012).
 - [9] J. M. Deutsch, Phys. Rev. A **43**, 2046 (1991).
 - [10] M. Srednicki, Phys. Rev. E **50**, 888 (1994).
 - [11] W. Beugeling, R. Moessner, and M. Haque, Phys. Rev. E **89**, 042112 (2014).
 - [12] A. Khodja, R. Steinigeweg, and J. Gemmer Phys. Rev. E **91**, 012120 (2015).
 - [13] C. Gogolin, Master thesis , The University Wurzburg, arXiv: 100.5058 (2010).
 - [14] K. R. Fratus and M. Srednicki, arXiv: 1505.04206 (2015).
 - [15] T. N. Ikeda, Y. Watanabe, and M. Ueda, Phys. Rev. E **87**, 012125 (2013).
 - [16] C. Gogolin, M. P. Muller, and J. Eisert Phys. Rev. Lett. **106**, 040401 (2011).
 - [17] Zelditch, Steve. ”Quantum ergodicity and mixing of eigenfunctions.” arXiv preprint math-ph/0503026 1 (2005): 183-196.
 - [18] R. Steinigeweg, A. Khodja, H. Niemeyer, C. Gogolin, and J. Gemmer, Phys. Rev. Lett. **112**, 130403 (2014).
 - [19] L. F. Santos, F. Borgonovi and F. M. Izrailev, Phys. Rev. Lett. **108**, 094102 (2012).
 - [20] T. A. Elsayed and B. V. Fine, Phys. Rev. Lett. **110**,

- 070404 (2013).
- [21] R. Steinigeweg, J. Gemmer, and W. Brenig, Phys. Rev. Lett. **112**, 120601 (2014).
 - [22] W. Beugeling, R. Moessner and Masudul Haque, Phys. Rev. E **89** 042112 (2014).
 - [23] H. De Raedt and K. Michielsen, *Computational Methods for Simulating Quantum Computers in Handbook of Theoretical and Computational Nanotechnology* (American Scientific Publishers, Los Angeles, 2006).
 - [24] K. De Raedt, K. Michielsen, H. De Raedt, B. Trieru, G. Arnold, M. Richter, T. Lippert, H. Watanabe, and N. Ito, Comput. Phys. Commun. **176**, 121 (2007).
 - [25] D. Rossini, A. Silva, G. Mussardo, and G. E. Santoro, Phys. Rev. Lett. **102**, 127204 (2009).
 - [26] M. Rigol, M. Srednicki, Phys. Rev. Lett. **108**, 110601 (2012).
 - [27] M. Rigol, Phys. Rev. Lett. **103**, 100403 (2009).
 - [28] T. N. Ikeda, Y. Watanabe, and M. Ueda, Phys. Rev. E **84**, 021130 (2011).
 - [29] H. Bethe, Z. Phys. **71**, 205 (1931).
 - [30] E. H. Lieb and W. Liniger, Phys. Rev. B. **130**, 1605 (1963).
 - [31] M. Gaudin Phys. Lett. A. **24**, 55 (1967).
 - [32] M. C. Gutzwiller, *Chaos in Classical and Quantum Mechanics* (Springer-Verlag, New York, 1990).
 - [33] T. A. Brody Lett. nuovo cimento . **7**, 482 (1973).
 - [34] C. Bartsch, and J. Gemmer, Phys. Rev. Lett **102**, 110403 (2009)
 - [35] M. Olshanii, Phys. Rev. Lett. **81**, 938 (1998)
 - [36] M. Rigol, V. Dunjko, and M. Olshanii, Nature **452**, 854 (2008).
 - [37] M. Rigol, Phys. Rev. Lett. **103**, 100403 (2009)
 - [38] R. Steinigeweg, J. Herbrych, and P. Prelovsek, Phys. Rev. E **87**, 012118 (2013)
 - [39] E. P. Wigner, Statistical distribution of the spacings of nuclear resonance levels, Camb. Phil. Soc: **47**, 790 (1951).
 - [40] F. J. Dyson, Statistical theory of the energy levels of complex systems, J. Math. Phys: **3**, 140 (1962).
 - [41] J. Gemmer, R. Steinigeweg and M. Michel Phys. Rev. B. **73**, 104302 (2006).
 - [42] J. Eisert, M. Friesdorf and C. Gogolin Nat. Phys. **11**, 124 (2015).
 - [43] P. Reimann Phys. Rev. Lett. **115**, 010403 (2015).
 - [44] A. I. Shnirelman Usp. Mat. Nauka **29**, 181 (1974).
 - [45] S. Dubey, L. Silvestri, J. Finn, S. Vinjanampathy, and K. Jacobs Phys. Rev. E **85**, 011141 (2012).
 - [46] A. Hutter and S. Wehner Phys. Rev. A **87**, 012121 (2013).
 - [47] B. S. Shastri J. Phys. A **44**, 052001 (2011).
 - [48] K.-J. Shi, S.-J. Chang *The Physics of Phase Space Non-linear Dynamics and Chaos Geometric Quantization, and Wigner Function*, Lect. Notes Phys. **278** (Springer, 130-132, 2005)
 - [49] P. Mazur Physica **43**, 533 (1969).
 - [50] C. Bartsch, R. Steinigeweg and J. Gemmer Phys. Rev. E **77**, 011119 (2008).